Two Degree of Freedom System Forced Vibration Theory

INTRODUCTION

Some dynamic systems that require two independent coordinates, or degrees of freedom, to describe their motion, are called "two degree of freedom systems". Degrees of freedom may or may not be in the same coordinate direction. Figure 1 (a) shows a system having two degrees of freedom in both the x and y direction. Figure 1 (b) shows another example where both degrees of freedom are in the same direction but on different masses.

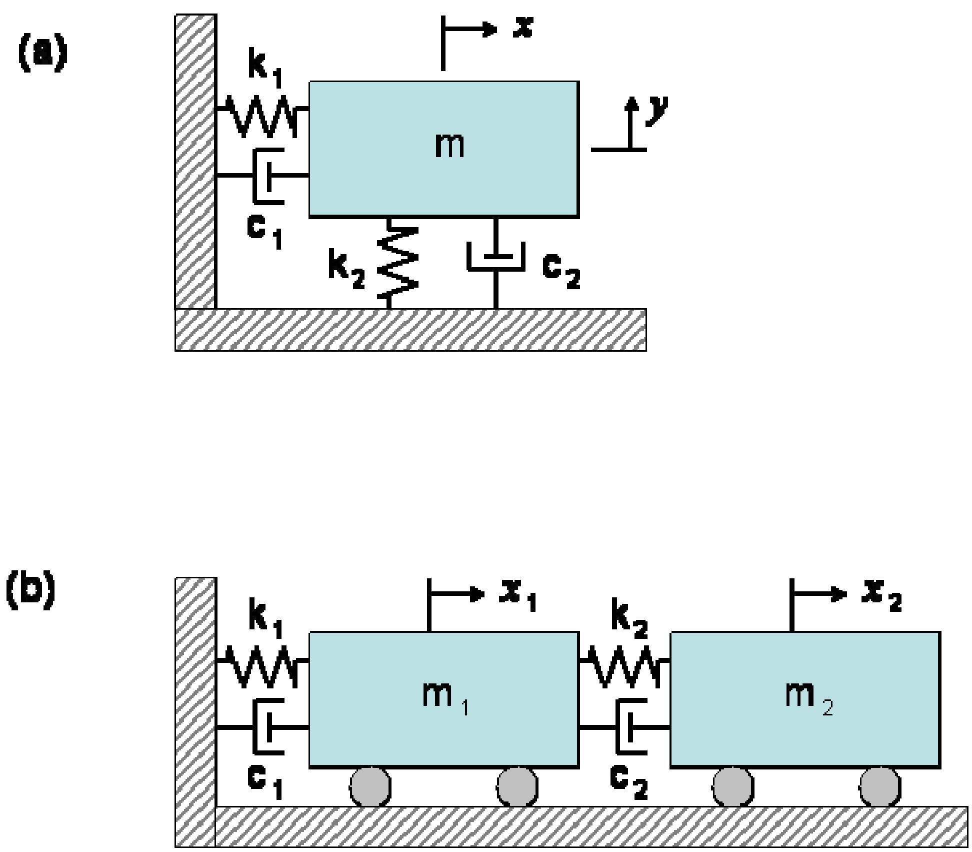

Figure 1: Examples of two degree of freedom systems. (a) Each degree of freedom shown here is in a different direction. (b) Multiple degree of freedom systems can also be in the same directions, but on different masses, as shown here.

For a two degree of freedom system there are two equations of motion, each one describing the motion of one of the degrees of freedom. In general, the two equations are in the form of coupled differential equations. Assuming a harmonic solution for each coordinate, the equations of motion can be used to determine two natural frequencies, or modes, for the system.

OBTAINING THE EQUATIONS OF MOTION

The equations of motion for a two degree of freedom system can be found using Newton's second law. Consider the system shown in Figure 1 (b). The coordinates that completely describe the motion of this system are $x_1(t)$ and $x_2(t)$, measured from the equilibrium position of each mass. External forces $F_1(t)$ and $F_2(t)$ act on masses m_1 and m_2 respectively. Using Newton's second law, we draw the free body diagrams of each mass as shown in Figure 2.

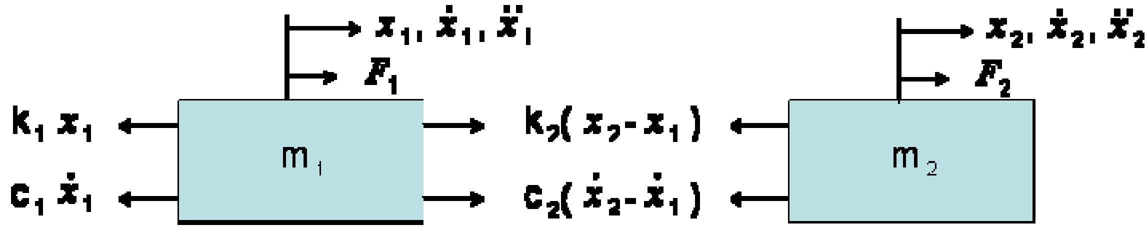

Figure 2: Free body diagrams for the masses in the two degree of freedom system illustrated in Figure 1 (b)

From these free body diagrams the equations of motion are easily found:

$$m_1 \cdot \ddot{x}_1 + (c_1 + c_2)\cdot \dot{x}_1 - c_2 \cdot \dot{x}_2 + (k_1 + k_2)\cdot x_1 - k_2 \cdot x_2 = F_1$$
$$m_2 \cdot \ddot{x}_2 - c_2 \cdot \dot{x}_1 + c_2 \cdot \dot{x}_2 - k_2 \cdot x_1 - k_2 \cdot x_2 = F_2$$

Note that both equations contain both coordinates x_1 and x_2, thus the differential equations are coupled. What this means physically is that the motion of one mass affects the motion of the other. It is useful to put the equations of motion into matrix form:

$$[m]\{\ddot{x}(t)\} + [c]\{\dot{x}(t)\} + [k]\{x(t)\} = \{F(t)\}$$

Where,

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$[c] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}$$

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

And,

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

The mass, damping, and stiffness matrices will always be square and of the dimension equivalent to the number of degrees of freedom, (which in this case is 2). It is also important to note that these matrices are typically symmetrical.

FORCED VIBRATION FOR A TWO DEGREE OF FREEDOM SYSTEM

For a general two degree of freedom system the equations of motion will be of the following form:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Assuming that the external forces are harmonic in nature, we represent them as :

$$F_j(t) = F_{j0} \cdot e^{i \cdot \omega \cdot t}, \quad j = 1, 2$$

and the steady-state solutions as :

$$x_j(t) = x_{j0} \cdot e^{i \cdot \omega \cdot t}, \quad j = 1, 2$$

X_1 and X_2 are complex values that are dependent of ω, and represent the dynamic characteristics of the system. The equations can now be put into terms of frequency as follows:

$$\begin{bmatrix} \left(-\omega^2 m_{11} + i \cdot \omega \cdot c_{11} + k_{11}\right) & \left(-\omega^2 m_{12} + i \cdot \omega \cdot c_{12} + k_{12}\right) \\ \left(-\omega^2 m_{12} + i \cdot \omega \cdot c_{12} + k_{12}\right) & \left(-\omega^2 m_{22} + i \cdot \omega \cdot c_{22} + k_{22}\right) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix}$$

Let mechanical impedance be :

$$Z_{rs}(\omega) = -\omega^2 \cdot m_{rs} + i \cdot \omega \cdot c_{rs} + k_{rs}, \quad r, s = 1, 2$$

The impedance matrix is defined as:

$$[Z(\omega)] = \begin{bmatrix} Z_{11}(\omega) & Z_{12}(\omega) \\ Z_{12}(\omega) & Z_{22}(\omega) \end{bmatrix}$$

and the system equations can be written as

$$[Z(\omega)]\{X(\omega)\} = \{F_0(\omega)\} \rightarrow \{X(\omega)\} = [Z(\omega)]^{-1}\{F_0(\omega)\} = [H(\omega)]\{F_0(\omega)\}$$

$$\{X(\omega)\} = \left[Z(\omega)\right]^{-1}\{F(\omega)\} = \left[H(\omega)\right]\{F(\omega)\}$$

$$\left[H(\omega)\right] = \left[Z(\omega)\right]^{-1}$$

The inverse of the impedance matrix is called the frequency response matrix $\left[H(\omega)\right]$.

The solution for each response is:

$$x_1(\omega) = H_{11}(\omega) F_1(\omega) + H_{12}(\omega) F_2(\omega)$$
$$x_2(\omega) = H_{21}(\omega) F_1(\omega) + H_{22}(\omega) F_2(\omega)$$

$H_{ij}(\omega)$ is a complex valued function of frequency and indicates the relationship between a response at degree of freedom i, and a force acting at degree of freedom j.

A typical frequency response function $H_{ij}(\omega)$ for a 2 degree of freedom system is shown below in magnitude / phase format.