Response of Second-Order Systems to Initial Conditions

INTRODUCTION

This tutorial discusses the response of a second-order system to initial conditions, including initial displacement and initial velocity. The mass-spring-dashpot system shown in Fig. 1 is an example of a second-order system. Other documents are also available which provide more specific details on second-order systems.

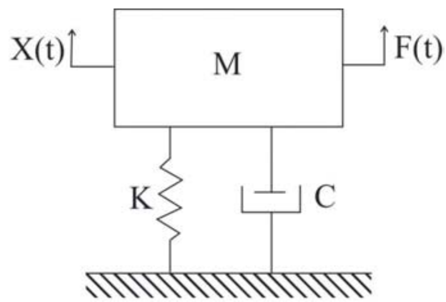

Fig. 1. Second-order mass-spring-dashpot system.

EQUATIONS DESCRIBING SYSTEM RESPONSE

The equation of motion describing the behavior of a second-order system is

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0 . \tag{1}$$

When a system is subjected to initial conditions, no force continues to act on the system after time t = 0. Therefore, the response to initial conditions is actually a free response. This response is found by solving for the homogeneous solution to the differential equation. The form of the response will depend on whether the system is under-damped, critically damped, or overdamped.

Under-damped

An underdamped system has damping less than critical damping, or $\zeta < 1$. For this case, the response to initial conditions is given by

$$x_p(t) = e^{-\sigma t}\left[x_0 \cos\omega_d t + \left(\frac{\sigma x_0 + v_0}{\omega_d}\right)\sin\omega_d t\right], \tag{2}$$

where

x_0 = initial displacement,

v_0 = initial velocity,

$\sigma = \zeta\omega_n$, and

$\omega_d = \omega_n\sqrt{1-\zeta^2}$.

Critically Damped

A critically damped system has damping equal to critical damping, or $\zeta = 1$. In this case, the response to initial conditions is given by

$$x_p(t) = x_0 e^{-\omega_n t} + (\omega_n x_0 + v_0) t e^{-\omega_n t}. \tag{3}$$

Over-damped

An over-damped system has greater than critical damping, or $\zeta > 1$. For this type of system, the response is given by

$$x_p(t) = \left[\frac{\omega_n \left(\zeta + \sqrt{\zeta^2 - 1} \right) x_0 + v_0}{2\omega_n \sqrt{\zeta^2 - 1}} \right] e^{-\omega_n \left(\zeta - \sqrt{\zeta^2 - 1} \right) t} - \left[\frac{\omega_n \left(\zeta - \sqrt{\zeta^2 - 1} \right) x_0 + v_0}{2\omega_n \sqrt{\zeta^2 - 1}} \right] e^{-\omega_n \left(\zeta + \sqrt{\zeta^2 - 1} \right) t}. \tag{4}$$

FORM OF SYSTEM RESPONSE

The following plots show the response of a second order system to varying initial conditions.

Initial Displacement

Recognizing an initial displacement is straightforward—the system's displacement does not begin at zero. Fig. 2 shows the response of three systems (under-damped, critically damped, and over-damped) to an initial displacement of 0.01. The damping ratios of the three systems are given in Table 1.

Table 1. Damping ratios for example systems.

System type	**Damping ratio (ζ)**
Under-damped	0.22
Critically damped	1.0
Over-damped	2.2

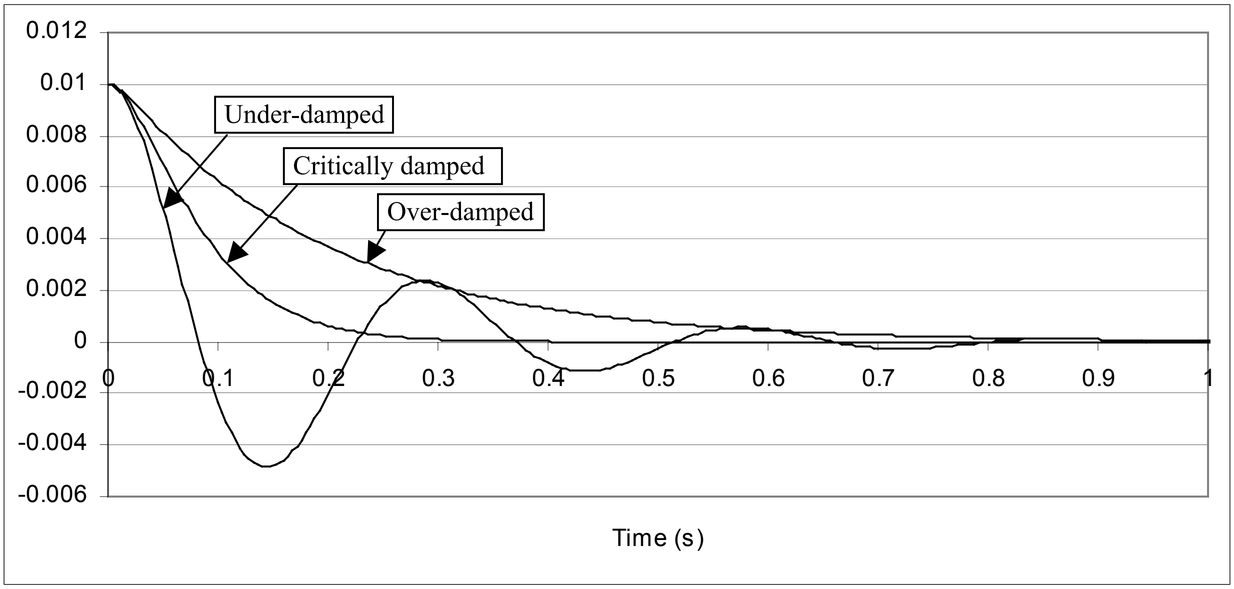

Fig. 2. Response of a second-order system to an initial displacement.

Initial Velocity

When a system is subjected to an initial velocity only, the response begins at zero and then goes up or down, depending on whether the system has been given a positive or negative initial velocity. The response to an initial velocity looks identical to that of an impulse, which is discussed in another tutorial. Fig. 3 shows the response of the same three systems to a positive initial velocity.

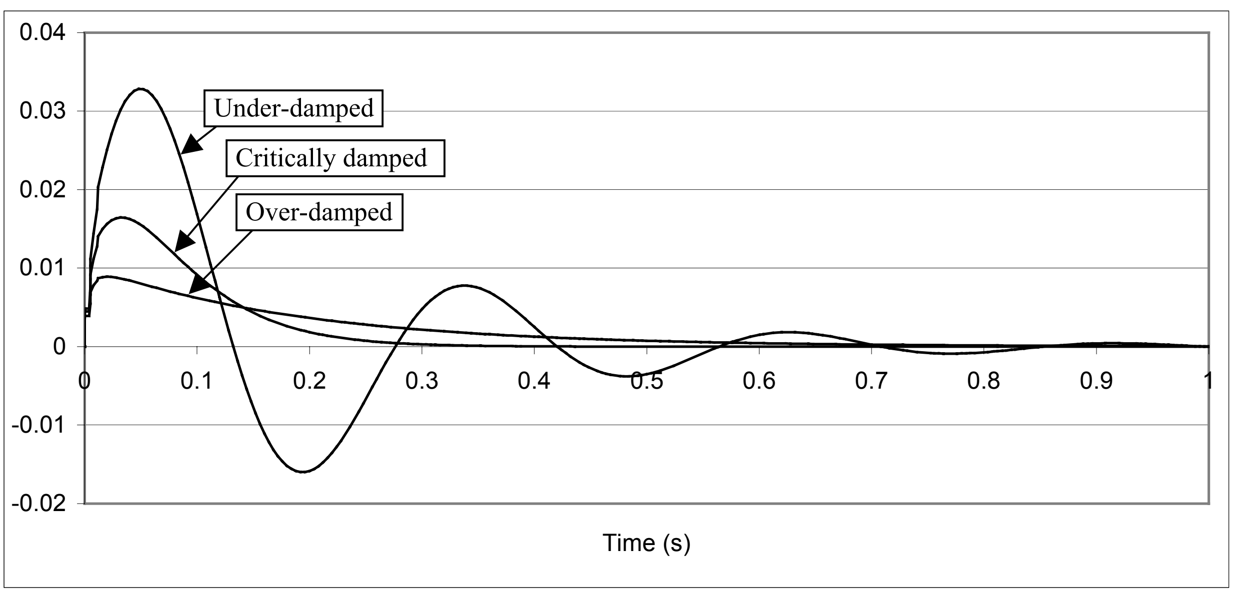

Fig. 3. Response of second-order system to a positive initial velocity.

Fig. 4 shows the response of these systems to a negative initial velocity. Note that this is identical to the plot in Fig. 3, except that it is reflected over the x-axis.

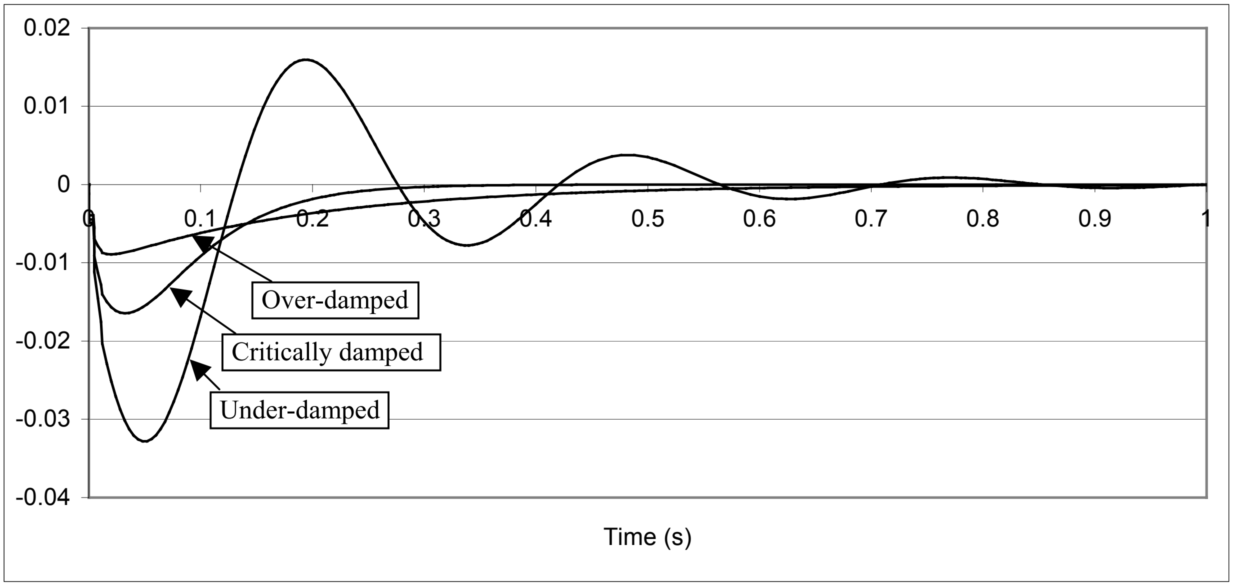

Fig. 4. Response of a second-order system to a negative initial velocity.

Combined Initial Displacement and Initial Velocity

Initial conditions can also be combined in various ways. Fig. 5 shows the response of the same three systems to an initial displacement of 0.01 and a positive initial velocity.

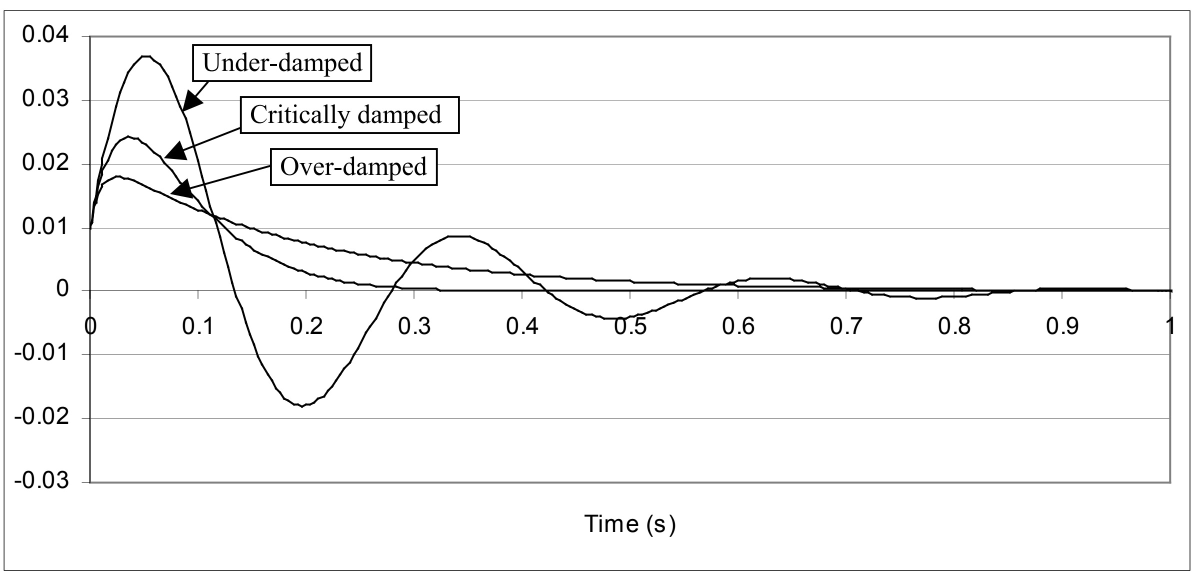

Fig. 5. Response of a second-order system to an initial displacement and a negative initial velocity.

Fig. 6 shows the response of the systems to an initial displacement of 0.01 and a negative initial velocity.

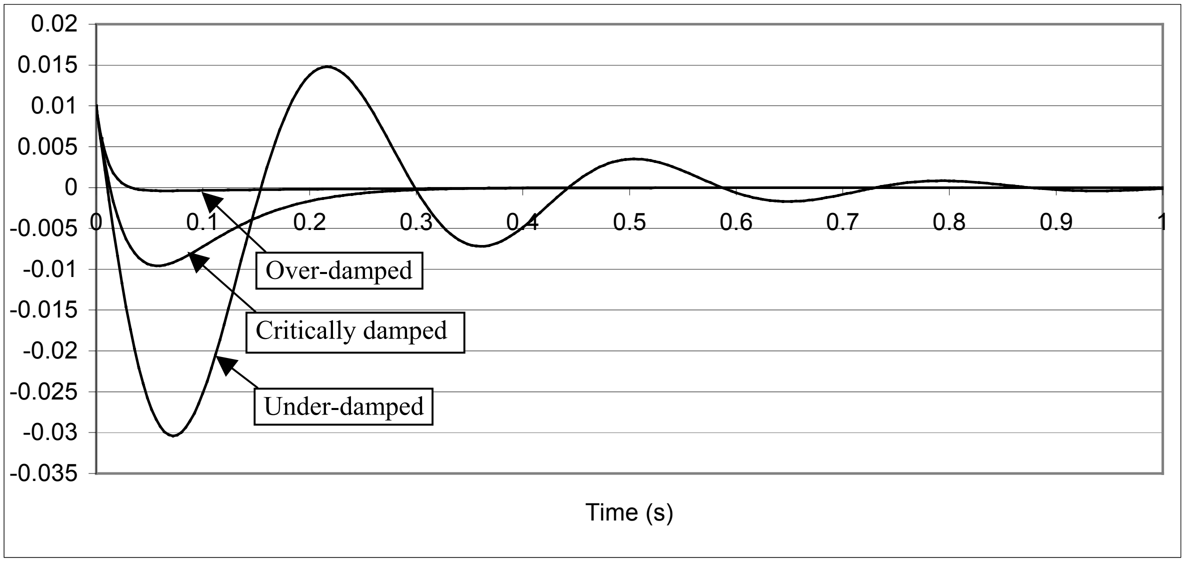

Fig. 6. Response of a second-order system to an initial displacement and a negative initial velocity.

Since an initial condition response is a free response, the steady state condition for all of these possibilities is zero.