Theory of First-Order Systems

INTRODUCTION

A first-order dynamic system is one whose behavior can be described with a first-order ordinary differential equation (ODE). A first-order ODE is one in which the highest-order derivative is a first derivative.

This type of problem includes tank-filling and mass-dashpot problems, such as those shown in Fig. 1.

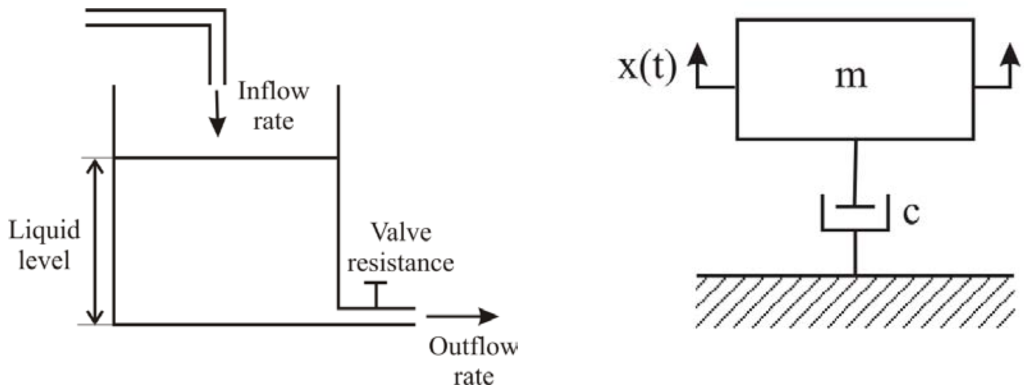

Fig. 1. First-order tank-filling problem and first-order mass-dashpot problem.

A simple RC circuit, like that shown in Fig. 2, also can be described with a first-order equation.

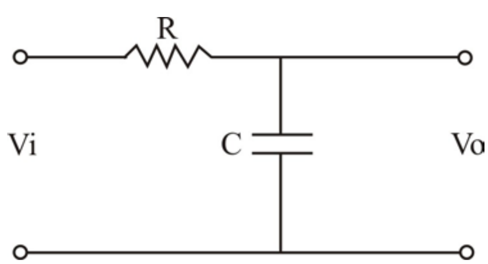

Fig. 2. First-order RC circuit.

The RC circuit is useful in that it can be used as a simple low-pass filter. It will be used as an example in this and other tutorials in this series.

SYSTEM EQUATION

In dynamic systems work, first order equations are typically written in the form

$$\dot{x} + \frac{1}{\tau} x = f(t), \tag{1}$$

where
x = displacement, or the equivalent property for the given system,
$\dot{x}$ = velocity, or equivalent,
τ = the time constant, and
$f(t)$ = the forcing function, a function of time.

The time constant, τ, is related to how fast or slow the system responds to an input or disturbance. The function on the right-hand side of the equation is the forcing function—some input to the system which is driving its response. In the case of the RC circuit, this would be the input voltage.

Equation (1) is in "standard form;" there is no constant multiplying the highest order derivative. It also has a constant coefficient—the value of $1/\tau$ does not vary with time.

For the RC circuit shown in Fig. 2, the governing equation is

$$\dot{v}_o + \frac{1}{RC} v_o = \frac{1}{RC} v_i . \qquad (2)$$

Note that instead of displacement and velocity (x and $\dot{x}$), there is the output voltage and the time rate of change of the output voltage (v_o and $\dot{v}_o$). Comparing (2) with (1) , it is seen that the time constant of the system is

$$\tau = RC . \qquad (3)$$

SYSTEM RESPONSE

The ODE (1) has homogeneous and particular solutions, x_h and x_p, which describe the response of the system. The general solution of (1) is given by $x = x_h + x_p$.

Homogeneous Solution

The homogeneous solution (x_h) depends on the inherent characteristics of the system and describes the system's free response. This is the solution of the equation

$$\dot{x} + \frac{1}{\tau} x = 0 , \qquad (4)$$

which is identical to (1) except that the right-hand side of the equation is equal to zero. This means that there is no forcing function; no force is continuing to act upon the system after time $t = 0$.

To find the system response, we first assume that the solution to this equation is in the form

$$x(t) = e^{\lambda t} . \qquad (5)$$

This is then substituted into the original equation, (4), to find

$$\lambda + \frac{1}{\tau} = 0 . \qquad (6)$$

This is the characteristic equation. Solving the characteristic equation for λ and then substituting the result back into (5), the solution becomes

$$x_h(t) = ce^{-t/\tau} , \qquad (7)$$

where c is a constant.

Particular Solution

The particular solution (x_p) to the ODE describes the response of the system when the right-hand side of (1), f(t), is non-zero. The form of the solution depends on the actual forcing function, which could be any time-varying function. Typical functions which have practical applications include impulses and step functions. These are discussed in detail in separate tutorials.

Once the particular solution (x_p) has been found, the general solution of the ODE is

$$x(t) = x_h + x_p = ce^{-t/\tau} + x_p \tag{8}$$

The initial state of the system, x(0), is assumed known. This determines the value of the constant c.