Filtering with RC Circuits

INTRODUCTION

This tutorial addresses the response of a first-order system, an RC circuit like that shown in Fig. 1, to a sinusoidal input signal.
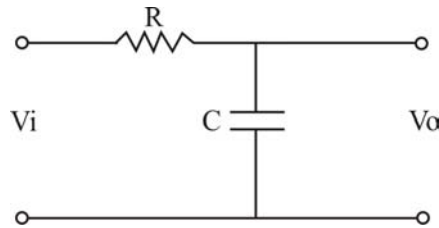

Fig. 1. RC Circuit

As will be shown, this circuit acts as a simple low-pass filter; it allows low-frequency sine waves to pass through relatively unaffected and attenuates high-frequency signals. The definition of "low-frequency" and "high-frequency" as they relate to an individual filter depends on the cut-off frequency of the filter, which will be discussed in detail.

This RC circuit is a first-order system. Another document is available which discusses the theory of first-order systems in more detail.

A Simulink model describing the behavior of this circuit will be used to demonstrate the effect of the circuit on various sine waves. Therefore, basic familiarity with Simulink is helpful but not necessary.

SYSTEM MODEL

The Simulink model in Fig. 2 represents a first-order, single-degree-of-freedom RC circuit. Note that the input to the system is a sine wave; this model will therefore show how the RC circuit will respond to a sinusoidal input such as an alternating electrical current.
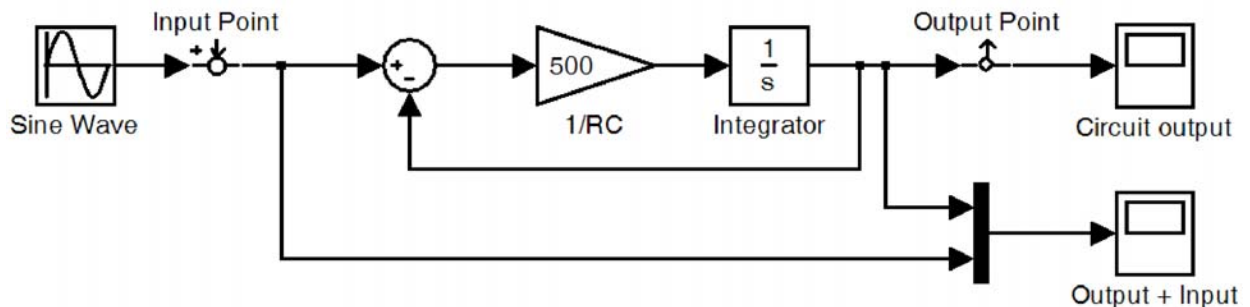

Fig. 2. RC circuit with sine wave input signal.

The system represented in Fig. 2 can also be described with a differential equation of the form

$$\dot{x} + \left(\frac{1}{RC}\right)x = f(t), \tag{1}$$

where
x = output voltage from the circuit,
$\dot{x}$ = the time rate of change of the output voltage,
R = the resistor value,
C = the capacitor value, and
f(t) = forcing function, a sine wave.

   As you can see in Fig. 2, $(1/RC) = 500$ for this circuit, so RC = 0.002. For this type of RC circuit, the value of RC is the time constant, τ, of the system. These numbers will resurface later on in this tutorial.

FILTERING CHARACTERISTICS

   Before examining the plots that were generated from the circuit model shown in Fig. 2, the behavior of the RC circuit will first be discussed.

   Bode Plot

   A Bode diagram will be used to better understand how an RC circuit affects an input voltage by displaying the characteristics of the circuit in the frequency domain. Fig. 3 shows the Bode diagram for two different RC circuits. The green line describes the circuit having an RC value of 0.01, and the blue line describes the circuit with an RC of 0.002.
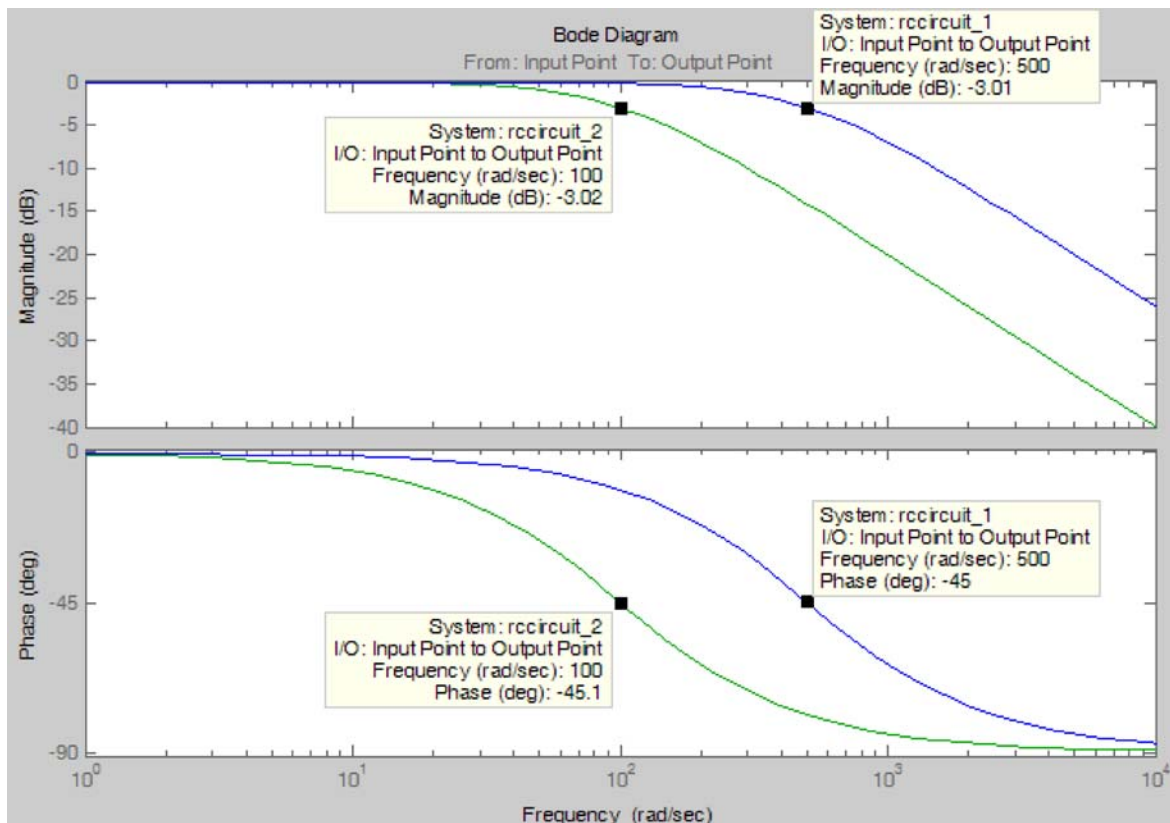

Fig. 3. Bode plots for RC circuits with different time constants.

A Bode Diagram actually consists of two separate plots. The top plot shows magnitude versus frequency on a log-log scale. This represents the magnitude attenuation of the signal—it gives the ratio of the output to the input in decibels. The magnitude of the ratio is always less than or equal to 1 because the input signal is either unaffected or attenuated—the output signal is never greater than the input. The bottom plot shows phase versus frequency on a linear-log plot, this represents the phase shift of the output with respect to the input. For both plots, the frequency is in rad/sec.

Cut-Off Frequency

In Fig. 3, the blue curves are for a circuit having a time constant of 0.002 seconds; this is the system shown in the model in Fig. 2. A tag has been placed on the blue line on the magnitude plot at $\omega = 500$ rad/sec. This location is called the "3dB down point" because at this frequency the magnitude is attenuated by 3 decibels. The attenuation is converted to decibels using

$$\text{magnitude(dB)} = 20\log_{10}\left(\frac{\text{output}}{\text{input}}\right). \tag{2}$$

Three dB is equivalent to an output/input ratio of $\sqrt{2}/2$, which is approximately equal to 0.707. This point is also called the cut-off or break frequency. The attenuation due to the filter gradually increases as the frequency increases, as can be seen by the smooth curve in the upper plot in Fig. 3. There is no clearly defined point representing the upper end of the frequency range of the filter. Therefore a point must be chosen, and the 3dB down point is frequently used. The reason for this can be seen in Fig. 4—if the line of the descending portion of the graph is extended back, the frequency at which it intersects the top of the graph is equal to the 3 dB down point.
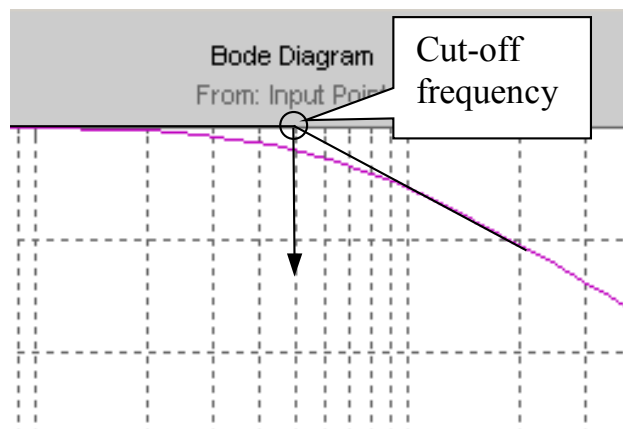

Fig. 4. Graphical method for determining cut-off frequency.

Signals with frequencies below the cut-off frequency are considered to be relatively unaffected by the filter. It should be noted, however, that a signal whose frequency is just below the cut-off will be attenuated by approximately 30% —this is not clear from the Bode diagram, as it is plotted on a log scale. Other types of filters can provide a sharper roll-off, but RC circuits have the advantage of being very simple and inexpensive to design and produce.

Cut-Off Frequency for RC Circuits

Recall that the time constant of the system (τ) shown in Fig. 2 is 0.002 seconds. Therefore $1/\tau$ is 500 rad/sec; this is the cut-off frequency of the circuit as was seen in Fig. 3. This is always true for first-order systems. To prove this, the equation describing the system will be examined further. The general equation for a first-order system is

$$\tau\dot{x} + x = f(t) \tag{3}$$

This is the same as (1), except that the equation has been rearranged slightly and RC has been replaced by τ. The transfer function (TF) of (3) is

$$TF \Rightarrow \frac{X(s)}{F(s)} = \frac{1}{\tau s + 1}. \tag{4}$$

The transfer function is found by taking the Laplace Transform of (3), assuming that the initial conditions are zero. The transfer function is the ratio of the Laplace transform of the output ($X(s)$) to the Laplace transform of the input ($F(s)$). To represent (4) as a frequency response function (FRF), $j\omega$ is substituted for s, where ω is frequency in rad/sec and j is $\sqrt{-1}$. The TF then becomes

$$\frac{X(j\omega)}{F(j\omega)} = \frac{1}{\tau(j\omega) + 1}. \tag{5}$$

The right side of the equation will now be modified. Multiply both the numerator and the denominator by the complex conjugate of the denominator to move the complex numbers to the numerator. The result is

$$\frac{X(j\omega)}{F(j\omega)} = \frac{1}{\omega^2\tau^2 + 1} - \frac{\omega\tau}{\omega^2\tau^2 + 1}j. \tag{6}$$

The magnitude of a number with real and imaginary parts is found using

$$\text{magnitude} = \sqrt{(Re)^2 + (Im)^2}. \tag{7}$$

Applying this to the right-hand side of (6), the result is

$$\text{magnitude} = \sqrt{\frac{1}{\left(\omega^2\tau^2 + 1\right)^2} + \frac{\omega^2\tau^2}{\left(\omega^2\tau^2 + 1\right)^2}}, \tag{8}$$

which simplifies to

$$\text{magnitude} = \sqrt{\frac{1}{\omega^2\tau^2 + 1}}. \tag{9}$$

Since it is known that the cutoff frequency, ω_c, occurs at magnitude $= \sqrt{2}/2$, this can be substituted into (9) to obtain

$$\frac{\sqrt{2}}{2} = \sqrt{\frac{1}{\omega_c^2\tau^2 + 1}}. \tag{10}$$

Solving for ω_c, the cut-off frequency is found to be

$$\omega_c = \frac{1}{\tau}. \tag{11}$$

Knowing this relationship greatly eases the design of an RC circuit to be used as a filter.

Phase Shift

The lower plot in the Bode diagram in Fig. 3 shows the phase shift of the output relative to the input. The phase shift in degrees is plotted versus the frequency in rad/sec. Notice that the phase shift varies from 0° to almost -90°. The negative sign of the phase shift means that the output lags behind the input.

FILTERING OF SINEWAVES

A sine wave signal will now be input into the RC circuit and the resulting output will be examined. The input and output will be compared in the time domain. Based on the previous discussion about cut-off frequency, it can be predicted that a sine wave with a frequency significantly lower than the cut-off frequency will have only a slight phase shift and small attenuation.

The first sine wave to be used has a frequency of 10 Hz. It is input into the system shown in Fig. 2 which has a cut-off frequency of 500 rad/sec, or approximately 80 Hz. Therefore 10 Hz would be considered a low frequency relative to the cutoff frequency. Fig. 5 compares the input and output signals using the scope block in the Simulink model.
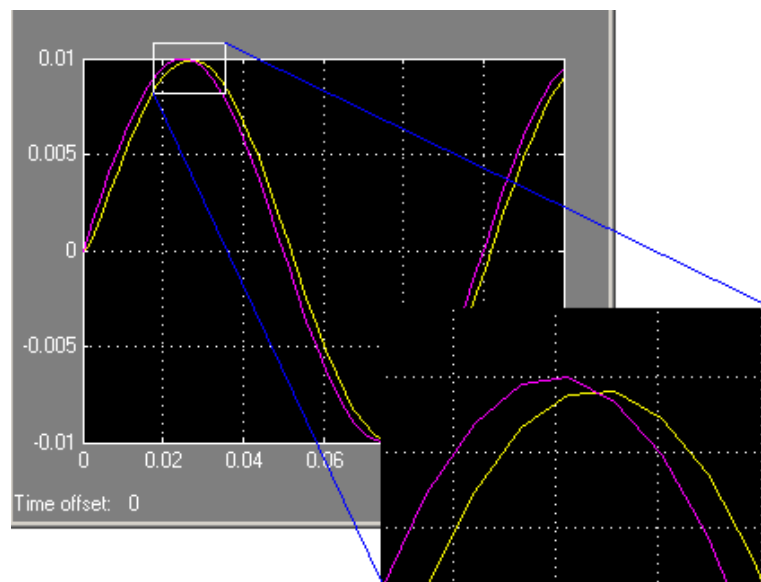

Fig. 5. Amplitude attenuation and phase shift of a 10 Hz sine wave.

We can see that the prediction was accurate, there is a small phase shift and some attenuation. To quantify the phase shift, the relative locations of the purple and yellow peaks are examined. The distance between peaks is 0.002 seconds, or approximately 7.2°. Looking on the Bode diagram in Fig. 6 at 10 Hz (62.8 rad/sec), it is found that the predicted phase shift is, as expected, 7.2°.
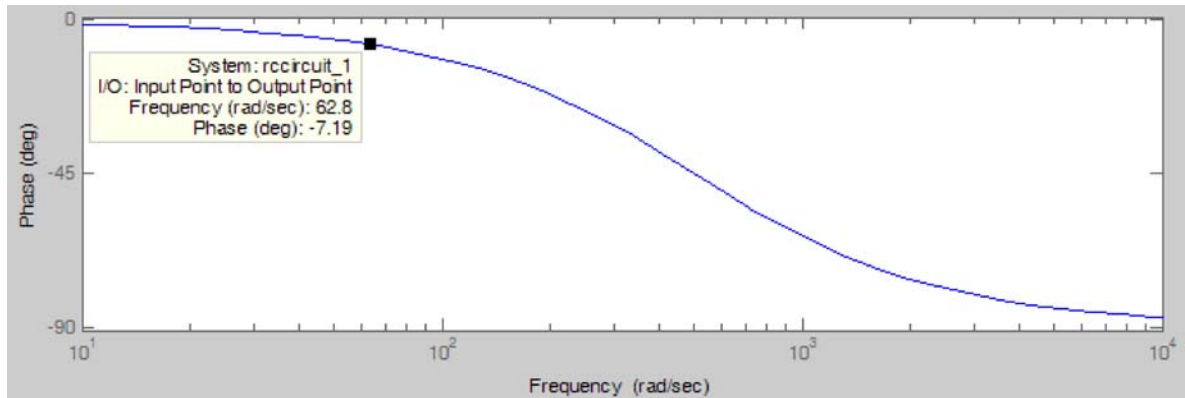
Fig. 6. Bode diagram showing phase shift at 10 Hz (62.8 rad/sec).

Considering the amplitude attenuation using similar means, by examining both the time-domain plot and the upper plot of the Bode diagram, it is found that the amplitude is attenuated by 0.069 dB.

Now a sine wave above the cut-off frequency will be examined. Fig. 7 shows a 200 Hz sine wave passed through the RC circuit.
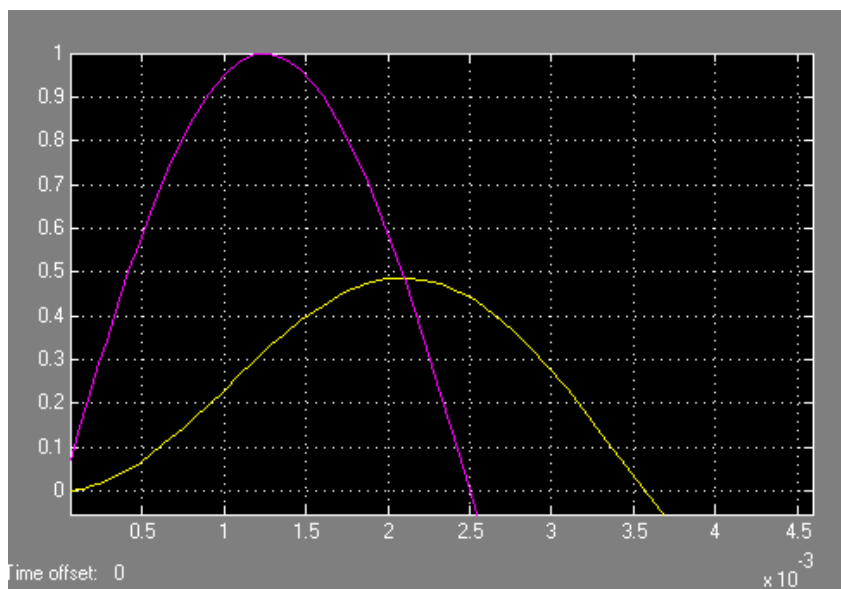

Fig. 7. Input versus output of a 200 Hz sine wave.

Clearly, this signal has been affected much more significantly than the previous example. For an input signal at 200 Hz, or 1257 rad/sec this filter results in a phase shift of −68.4° and an amplitude attenuation of 8.64 dB.